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# A method for calculating the stress–strain state in the general boundary value problem of metal forming—part 2. Impact of a bar against a rigid obstacle

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**Abstract**

The method for calculating stress–strain state and fracture proposed by Kolmogorov (1995) and in Part 1 of this present paper is illustrated by the simple problem of a thin bar impacting a rigid obstacle. Known exact solutions are used to test the method. On the basis of the stability theory, the one-dimensional solution has been shown to be legitimate. Mathematical simulation of bar fragmentation resulting from impact has been carried out. 

## 1. Calculation of stress–strain state on elastic bar impact

Suppose a thin bar of length $L$ moves at a rate $v_*$ and at $t = t_0 = 0$ begins to interact with a rigid obstacle (Fig. 1). The stress and strain states for $t > t_0$ need to be defined. The bar is assumed to be incompressible and isotropic. Besides, the mass forces (excepting inertial) are assumed to be negligible, and the constitutive equations (physical coupling equations) for deviators are represented by some functionals. In the subsequent discussion, the Lagrangian variables $x$, $0 \leqslant x \leqslant L$ will be used. As a Lagrangian coordinate we take the coordinates of the particles at the instant $t = t_0$, and in the problem discussed in this section they will coincide with Eulerian ones, as the deformations are small.

In our problem, the variational equation for the principle of virtual velocities and stresses (see e.g. Kolmogorov, 1995; Kolmogorov, 1986) at an arbitrary fixed instant of time has the form

$$\delta I = \delta \left\{ \int_0^L (\sigma_{ij}\xi'_{ij} + \rho w_i v'_i)\,\mathrm{d}x \right\} = 0. \tag{1}$$

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Here $I$ is the functional of the principle: $\sigma_{ij}$, $\xi_{ij}$ are stress tensor and strain rate tensor components; $v_i$, $w_i$ are velocity and acceleration components; $\rho$ is density. Repeated indices indicate summation for possible values $x$, $y$, $z$. Varying quantities are marked with a prime sign.

Let the constitutive relations be represented by the Hooke law, which, on account of the incompressibility of the material acquires the form of the functional

$$s_{ij} = 2G\int_0^t \xi_{ij}\,\mathrm{d}t, \tag{2}$$

where $G$ is modulus of shear, $G = E/3$, $E$ is Young's modulus.

### 1.1. *Solution by the variational-difference method*

Solve the variational problem (1) by the difference model. Choose the one-dimensional virtual fields of velocities and stresses as:

$$v'_x = v' = \frac{v'_i - v'_{i-1}}{\Delta}(x - x_{i-1}) + v'_{i-1}, \quad x_{i-1} \leqslant x \leqslant x_i; \tag{3}$$

$$\sigma'_{xx} = \sigma' = \frac{\sigma'_i - \sigma'_{i-1}}{\Delta}(x - x_{i-1}) + \sigma'_{i-1}, \quad x_{i-1} \leqslant x \leqslant x_i, \tag{4}$$

the rest of velocity and stress components being equal to zero. Here: $x_0 = 0, x_1, \ldots, x_n = L$ are the coordinates of even segmentation points within $[0, L]$; $\Delta = L/n$; $v'_i = v_i(t)$ and $\sigma'_i = \sigma_i(t)$ are the unknown values of velocity and stress in the segmentation nodes. The equation of equilibrium $\partial\sigma'/\partial x = \rho w$ is satisfied by entering it in the functional (1) with the Lagrange multiplier $v$, which is also represented by the piecewise linear function

$$\nu' = \frac{\nu'_i - \nu'_{i-1}}{\Delta}(x - x_{i-1}) + \nu'_{i-1}, \quad x_{i-1} \leqslant x \leqslant x_i. \tag{5}$$

Taking into account that

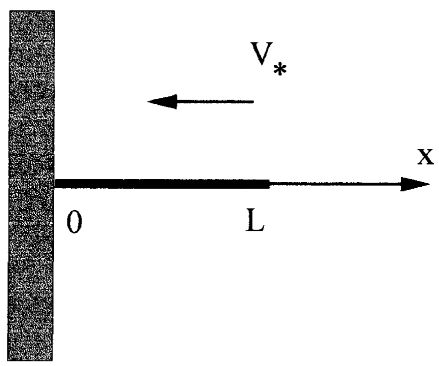


Fig. 1. Impact of a bar against a rigid obstacle.

$$\xi'_{xx} = \frac{v'_i - v'_{i-1}}{\Delta}, \quad x_{i-1} \leqslant x \leqslant x_i, \tag{6}$$

the functional of principle (1) has the form of the function of the variables $v'_i$, $\sigma'_i$, $v'_i$, $i = 0, \ldots, n$:

$$I = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left[ \sigma_{xx} \frac{v'_i - v'_{i-1}}{\Delta} + \rho w \left( \frac{v'_i - v'_{i-1}}{\Delta}(x - x_{i-1}) + v'_{i-1} \right) + \left( \frac{v'_i - v'_{i-1}}{\Delta}(x - x_{i-1}) + v'_{i-1} \right) \left( \frac{\sigma'_i - \sigma'_{i-1}}{\Delta} - \rho w \right) \right] \mathrm{d}x. \tag{7}$$

According to the method from Kolmogorov (1995) and Part 1 of the present paper, by substituting (after differentiation) the relations

$$v_i = \mathrm{d}u_i/\mathrm{d}t, \quad w = \frac{\mathrm{d}^2 u_i/\mathrm{d}t^2 - \mathrm{d}^2 u_{i-1}/\mathrm{d}t^2}{\Delta}(x - x_{i-1}) + \mathrm{d}^2 u_{i-1}/\mathrm{d}t^2 \tag{8}$$

and the value

$$\sigma_{xx} = \frac{3}{2} s_{xx} = 3G \frac{(u_i - u_{i-1})}{\Delta}, \quad x_{i-1} \leqslant x \leqslant x_i, \tag{9}$$

corresponding to the constitutive relations (2), into the necessary extremum conditions of function (7)

$$\partial I / \partial v'_i = 0, \quad \partial I / \partial \sigma'_i = 0, \quad \partial I / \partial v'_i = 0. \tag{10}$$

we obtain the sets of differential equations for finding the values of $u_i(t) = \int_0^t v_i(\tau)\,\mathrm{d}\tau$—displacements of the nodes relative to the initial position and $\sigma_i(t)$:

$$A\,\mathrm{d}^2 u/\mathrm{d}t^2 = Bu, \tag{11}$$

$$D\sigma = F\,\mathrm{d}^2 u/\mathrm{d}t^2, \tag{12}$$

where $u = (u_0, u_1, \ldots, u_n)^T$, $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_n)^T$ are vector-functions of time; A, B, D, F are the constant matrices.

As an example, the set (11) has been solved numerically (by the third-order Runge–Kutta method) with the initial conditions: $u_i(t_0) = 0$, $i = 0, \ldots, n$; $v_i(t_0) = \mathrm{d}u_i/\mathrm{d}t = v_*$, $i = 1, \ldots, n$, $v_0(t_0) = 0$, for $n = 10$, with the following values of the parameters: $\rho = 8000$ kg/m$^3$; $L = 0.1$ m; $E = 200{,}000$ MPa. The values of displacements and stresses in the nodes as functions of time have been found, as well as the time of bar rebound $t_* = 0.00004$ s [from the condition $\sigma_0(t_*) = 0$] required for a new boundary value problem statement: bar motion after the rebound if fracture does not occur. The problem of the post-rebound motion was solved similarly. Note that the following boundary conditions are taken into account in solving: $u_0 = v_0 = \sigma_n = 0$ in the stage of bar–obstacle interaction, and $\sigma_0 = \sigma_n = 0$ after the rebound. As initial conditions for the second stage, the values of node stresses, displacements and velocities obtained at the instant of rebound are taken. The displacements and velocities for $n = 10$ agree well with the exact solutions to the wave equation describing the longitudinal vibrations of an elastic bar, while stresses require finer segmentation or a different approximation.

### 1.2. *Solution with Fourier series*

Now we solve the same variational problem (1) by using another interpolation of unknown functions, namely, find displacement velocities $v(t,x)$, stresses $\sigma(t,x)$ and the Lagrangian multiplier $v(t,x)$ for the equation of equilibrium as segments of the trigonometric Fourier series with known coordinate functions satisfying the boundary conditions $v'(t,0) = \sigma'(t,L) = 0$ and the unknown coefficients $a_i(t)$, $b_i(t)$, $c_i(t)$. For the stage of interaction, the virtual fields have the form:

$$v'(t,x) = \sum_{i=1}^{n} a_i(t)\sin(\alpha_i x), \tag{13}$$

$$\sigma'(t,x) = \sum_{i=1}^{n} b_i(t)\cos(\alpha_i x), \tag{14}$$

$$v'(t,x) = \sum_{i=1}^{n} c_i(t)\sin(\alpha_i x), \tag{15}$$

where $\alpha_i = \pi(i-0.5)/L$.

By using the method proposed by Kolmogorov (1995), as was done in Section 1.1, to find the unknown coefficients, we obtain the uniform set of linear ordinary differential second-order equations:

$$a_i = \frac{\mathrm{d}d_i}{\mathrm{d}t}, \quad \frac{\mathrm{d}^2 d_i}{\mathrm{d}t^2} = -\frac{3G}{\rho}\alpha_i^2 d_i, \quad b_i = -\frac{\rho}{\alpha_i}\frac{\mathrm{d}^2 d_i}{\mathrm{d}t^2}, \tag{16}$$

the solution result being

$$u(t,x) = \sum_{k=1}^{n} A_k \sin(a\alpha_k t)\sin(\alpha_k x), \tag{17}$$

where $a = \sqrt{3G/\rho}$ and the coefficients $A_k$ are found from the initial conditions

$$u(0,x) = f(x), \quad 0 \leqslant x \leqslant L \tag{18}$$

as follows

$$A_k = (2a/L\alpha_k)\int_0^L f(x)\sin(\alpha_k x)\,\mathrm{d}x. \tag{19}$$

As the function $f(x)$, we can choose the continuous function approximating the discontinuous initial conditions of the impact problem:

$$u_{,t}(0,0) = 0, \quad u_{,t}(0,x) = v_*, \quad 0 < x \leqslant L. \tag{20}$$

The solution (17) coincides with the sum of the first $n$ members of the series presenting the exact solution for the wave equation.

Similarly, the problem is solved for bar motion after rebound. The virtual fields satisfying the boundary conditions $\sigma'(t,0) = \sigma'(t,L) = 0$ were taken as follows

$$v'(t,x) = \sum_{i=0}^{n} a_i(t)\cos(\beta_i x), \tag{21}$$

$$\sigma'(t,x) = \sum_{i=0}^{n} b_i(t)\sin(\beta_i x), \tag{22}$$

$$v'(t,x) = \sum_{i=0}^{n} c_i(t)\cos(\beta_i x), \tag{23}$$

where $\beta_i = \pi i/L$. As the initial conditions for the second stage, the values of displacement, velocity and stress by the instant of rebound are taken. Figure 2 shows the value stresses in the nodes calculated by means of the series for $v_* = 250$ m/s, $n = 20$.

The solution in the form of series segments gives a more exact result than the variational-difference one (particularly, with respect to stresses), however it is not always possible to select coordinate functions.

Problem solutions in Sections 1.1 and 1.2 are necessary for testing the method and they are based on the assumption that the bar does not fracture (there is no macrofragmentation). Now we turn to the prediction of microdamage and macrofragmentation according to Kolmogorov (1995) and Part 1 of the paper.

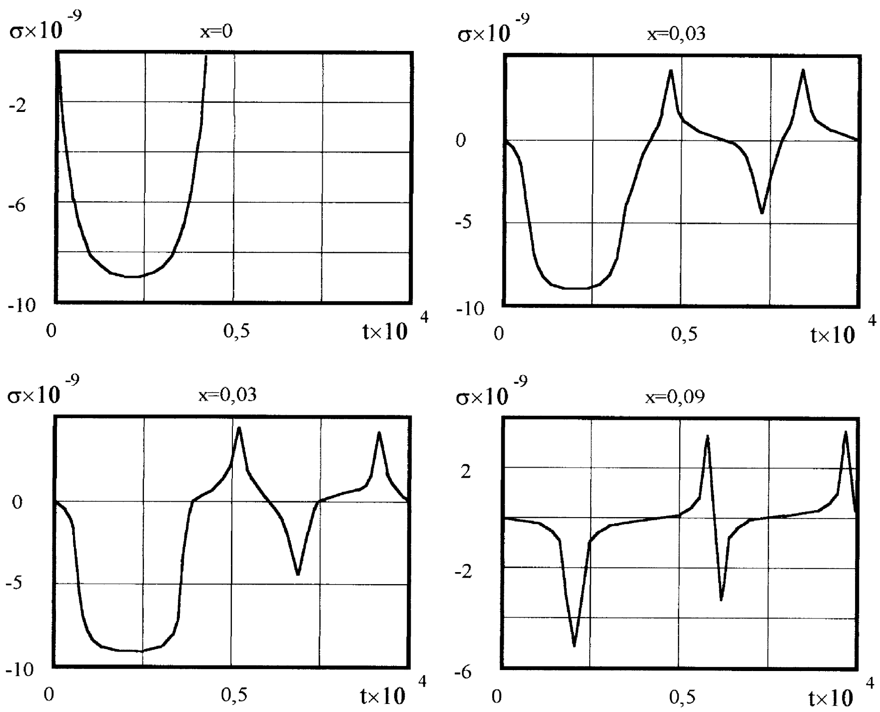


Fig. 2. Solution with the Fourier series. Stress.

### 1.3. *The calculation of bar damage*

According to Kolmogorov (1995) and Part 1 of the paper, for the problem solutions obtained, we have calculated damage caused by vibrations (due to fatigue) in the course of bar motion in order to define the times, $t_p$ and coordinates $x$ of microfractures.

Damage, $\psi$, is predicted for a material particle as follows. Sections of monotonic deformation are singled out on the motion path. Within the section, the strain rate $\xi_{xx}$ does not change its sign. We indicate by $t_1, t_2, \ldots, t_{n-1}$—the instants of $\xi_{xx}$ sign change. On the $n$-th section $t_{n-1} \leqslant t < t_n$

$$\psi(t) = \sum_{i=1}^{n} [\psi_i(t_i)]^{\alpha_i},$$

$$\frac{d\psi_n}{dt} = \frac{H(t)}{\Lambda_p[k_1(t), k_2(t)]}, \quad \psi_n(t_{n-1}) = 0. \tag{24}$$

Here $H = H(t)$ is shear strain rate intensity, $k_1 = \sigma/T$, $k_2 = 2(\sigma_{22}-\sigma_{33})/(\sigma_{11}-\sigma_{33})-1$, $\sigma$ is mean normal stress, $T$ is tangential stresses intensity, $\sigma_{11} \geqslant \sigma_{22} \geqslant \sigma_{33}$ are principal normal stresses, $\Lambda_p = \Lambda_p(k_1, k_2)$ is plasticity, $\alpha_i = \alpha_i(\bar{k}_1, \bar{k}_2)$ are the values of the function $\alpha = \alpha(k_1, k_2)$ in the $i$-th region of monotonic deformation. By the instant of fracture $(t = t_p)$ $\psi(t) = \psi(t_p) = 1$.

The functions $\Lambda_p$ and $\alpha$ are taken from Bogatov et al. (1984):

$$\Lambda_p = \chi \exp(\lambda\sigma/T), \quad \alpha = \alpha_0 \exp(1+0.238\sigma/T) \tag{25}$$

with the following material constants: $\chi = 0.2$, $\lambda = -2$, $\alpha_0 = 1.2$. For simplicity, damage is calculated for the finite number of points $y_i = iL/0.01$, $i = 1, \ldots, 100$. The simplification error does not exceed the solution error. In the models under study, the value of damage reaches one in several points at a time, to be more exact, on some segments where macrofracture will occur. Figure 3 shows damage distribution in the bar at the instant when the first macrorupture occurs $t_p = 0.000043$ s at the impact velocity of 250 m/s. It should be noted that, in all the model experiments, the distribution of damage, $\psi$, along the bar was seen to be an oscillating function.

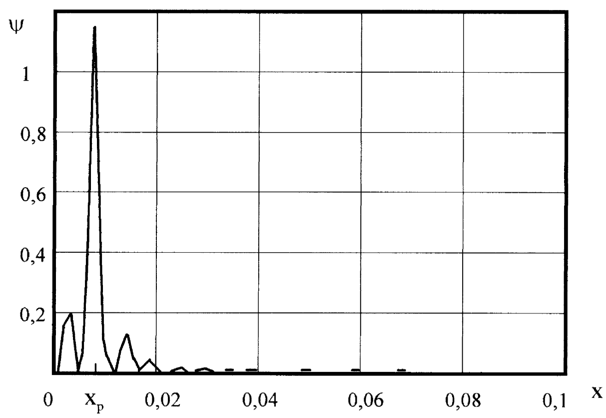


Fig. 3. Damage throughout the length of elastic bar.

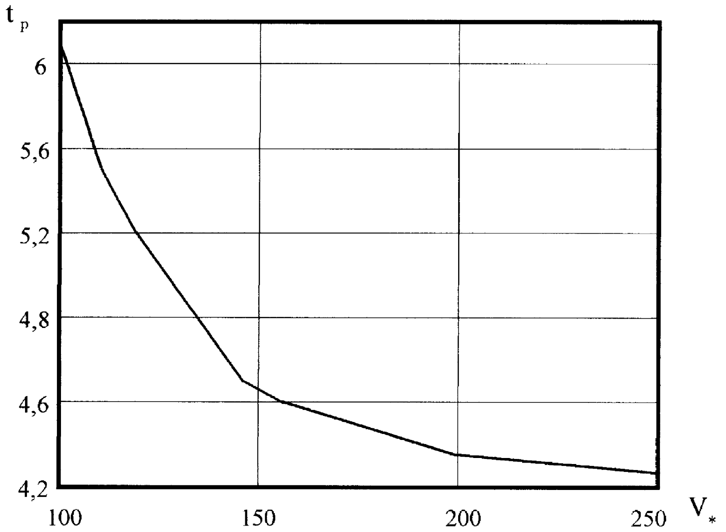


Fig. 4. The instant of the first fracture.

Figures 4 and 5 show the time of fracture, $t_p$, and the rupture point coordinates $x_p$ as dependent on impact velocity, $v_*$. Similarly, the subsequent times and points of rupture can be predicted. Figure 6 shows the motion of bar fragments after the first rupture.

Post-impact solid fragmentation caused by material fatigue in vibration is known from experiments. For example, this mechanism may be responsible for the failure of Rupert's glass drops (see Johnson and Chandrasekar, 1992).

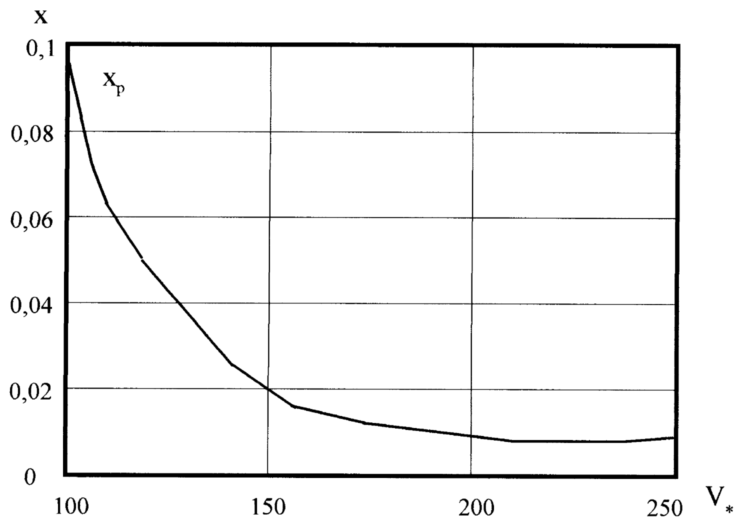


Fig. 5. The point of the first fracture.

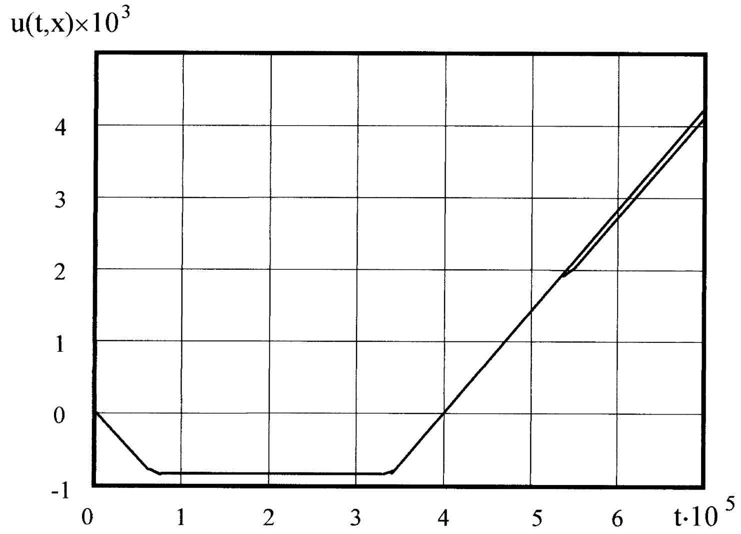


Fig. 6. Fracture point displacement.

The authors are aware of the fact that the method proposed in Part 1 of the paper and Kolmogorov (1995) and illustrated here should be supplemented with the description of dissipation of mechanical energy into heat energy and the description of vibration damping due to internal friction and external resistance.

## 2. Plastic bar impact on a rigid obstacle

Consider plastic bar impact when the bar takes only residual, namely, large strains. The material is assumed to be incompressible, the constitutive relation having the form:

$$T = Q\left(\int_0^t H\,\mathrm{d}t\right)^{\alpha}, \quad 0 \leqslant \alpha < 1. \tag{26}$$

Here we deal with large strains and the history of the Lagrangian coordinate system needs to be taken into account. For the chosen type of strain (3), the nonzero components of the metric tensor are as follows

$$g_{xx} = \frac{1}{g^{xx}} = \frac{1}{(1+(u_i - u_{i-1})/\Delta)^2},$$
$$g_{yy} = g^{yy} = g_{zz} = g^{zz} = 1, \quad x \in [0, L]. \tag{27}$$

They do not depend on $x$, therefore all the Cristoffel symbols are equal to zero, whereas the covariant derivative used in the equilibrium equations and kinematic equations, as in Part 1 of the paper, is equal to the corresponding partial derivative:

$$\nabla_x = \partial/\partial x. \tag{28}$$

In this case the functional of the virtual velocities and stresses principle can be written as follows

$$I = \int_0^L (TH' + \rho w v')\,\mathrm{d}x. \tag{29}$$

Note that $v'$ is a covariant velocity component, $w$ is a contravariant acceleration component expressed via the metric tensor as follows

$$w = g^{xx}\frac{\mathrm{d}v}{\mathrm{d}t} = \left(1 + \frac{(u_i - u_{i-1})}{\Delta}\right)^2 \frac{\mathrm{d}v}{\mathrm{d}t}. \tag{30}$$

We choose the virtual fields in the difference form (3)–(5). Taking into account that

$$v'_{i-1} > v'_i \quad \text{and} \quad H' = \sqrt{3}\frac{|v'_i - v'_{i-1}|}{\Delta} = \sqrt{3}\frac{v'_{i-1} - v'_i}{\Delta},$$

the functional takes the form of the function

$$I = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left[ T\sqrt{3}\frac{|v'_i - v'_{i-1}|}{\Delta} + \rho w \left(\frac{v'_i - v'_{i-1}}{\Delta}(x - x_{i-1}) + v'_{i-1}\right) + \left(\frac{v'_i - v'_{i-1}}{\Delta}(x - x_{i-1}) + v'_{i-1}\right)\left(\frac{\sigma'_i - \sigma'_{i-1}}{\Delta} - \rho\omega\right)\right]\mathrm{d}x. \tag{31}$$

By substituting eqn (30) and the relation

$$T = Q\left(\int_0^t \sqrt{3}\frac{v_{i-1} - v_i}{\Delta}\,\mathrm{d}t\right)^\alpha = Q\left(\frac{\sqrt{3}(u_i - u_i)}{\Delta}\right)^\alpha, \quad x \in [x_{i-1}, x_i], \tag{32}$$

into the necessary extremum conditions (10), we obtain a set of equations consisting of eqn (12) and

$$L(u)\,\partial^2 u/\partial t^2 = K(u) \tag{33}$$

to find the functions $u_i(t)$ and $\sigma_i(t)$. Here $L(u)$, $K(u)$ are nonlinear vector functions of the vector argument; $u$, $\sigma$, $D$, $F$ are the same as Section 1.1.

Set (33) has been solved numerically (by the third-order Runge–Kutta method) with the initial conditions and parameter values from Section 1.1, except $v_* = 300$ m/s, $Q = 2000$ and $\alpha = 0.5$. The calculation results are shown in Fig. 7.

Bar damage was calculated for the obtained solution. Table 1 shows the values of $\psi$ in each of the 10 segments $[x_{n-1}, x_n]$, with the impact velocity of 300 m/s, $t_p = 0.000015$ s. The first fracture occurs in the first element.

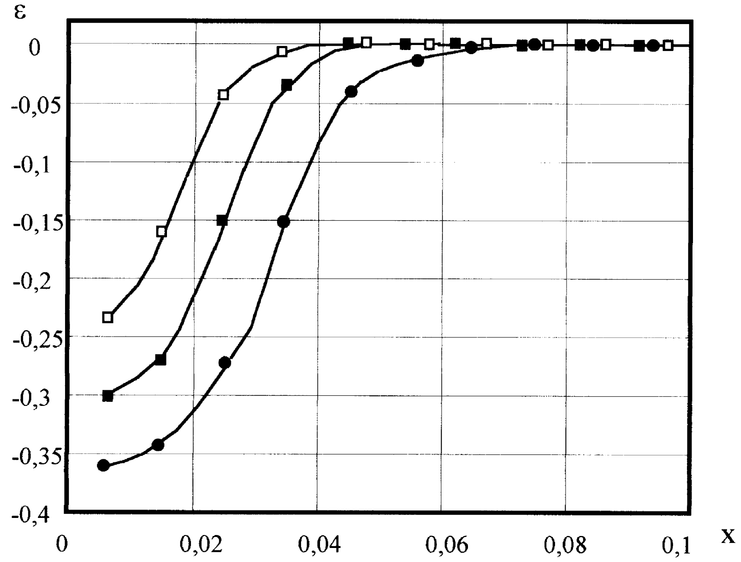


Fig. 7. The strain of the plastic bar before fracture: □, $t = 0.00001$; ■, $t = 0.00002$; •, $t = 0.000033$.

Table 1
Damage along the elastoplastic bar

| Element No. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Damage, $\psi$ | 1.002 | 0.878 | 0.623 | 0.199 | 0.059 | 0.020 | 0.009 | 0.004 | 0.004 | 0.007 |

## 3. Stability of the rectilinear shape of the elastic bar impacting a rigid obstacle

The method proposed in Part 1 and Komogorov (1995) offers approximate solution for the problem of motion stability of systems with distributed parameters. One example is discussed below.

In the above, one-dimensional solutions for the problem of thin bar impact on a rigid obstacle was obtained. However, common sense and experience are suggestive of the fact that the bar may have lateral displacements as well. Therefore the assumption of the above-described deformation type may prove unrealizable in practice or realizable under certain conditions. The stability of lateral displacements of bar points in impact is discussed here. Particularly, the rate of growth of lateral displacements is estimated: if it is low under some conditions, then the solutions given in Sections 1 and 2 can be considered realizable in practice under these conditions.

Applying the above-mentioned variational principle, we obtain the known equations of plane lateral–longitudinal vibrations of an elastic bar (see e.g. Berezovsky, 1976). The boundary conditions are as follows. At the tangency point, the bar end displacements and the bending moment are zero. At the free end, the lateral force, bending moment and longitudinal force are zero. Note that these boundary conditions allow for the displacement of the bar as a unit (rotation about the tangency point) and are called irregular. In the literature, as a rule, cases of regular fixation are discussed.

Since the problem under study is that of stability, the bar deflection from the long axis (abscissa) and the angle of rotation (derivative of deflection with respect to the abscissa) are considered small. After the linearization, the equation of longitudinal vibrations becomes independent of the equation of lateral vibrations and it is integrated independently. The solution is well-known and it is constructed in the form of a trigonometric series. As a result, after substituting this solution into the equation of lateral vibrations we obtain a parabolic equation with periodic coefficients.

The solution for this equation is pursued in the form of the series:

$$u(\tau,\xi) = \sum_{m=0}^{\infty} \varphi_m(\xi) w_m(\tau). \tag{34}$$

Here $\varphi_m(x)$ are eigenfunctions of the boundary value problem

$$\frac{d^4\varphi_m}{d\xi^4} + q_m^4 \varphi_m = 0, \tag{35}$$

$$\left.\frac{d^2\varphi_m}{d\xi^2}\right|_{\xi=0} = 0, \quad \left.\frac{d^3\varphi_m}{d\xi^3}\right|_{\xi=0} = 0, \quad \varphi_m(1) = 0, \quad \left.\frac{d^2\varphi_m}{d\xi^2}\right|_{\xi=1} = 0 \tag{36}$$

having the form

$$\varphi_m(\xi) = sh(q_m(\xi-1))/ch(q_m) + \sin(q_m(\xi-1))/\cos(q_m), \quad m = 1,2,3.$$

$\varphi_0(\xi) = \xi - 1$. The numbers $q_m$ satisfy the transcendental equation $tg(q_m) = th(q_m)$. This equation has a countable number of roots, including the double zero root corresponding to the rotation of the bar as a solid about the point of contact with the obstacle. One eigenfunction corresponds to this root. The roots asymptotically approach the numbers of the form $\pi/4 + \pi n$, $n \to \infty$.

Substitution and rearrangement give a countable set of ordinary differential equations with periodic coefficients with respect to the functions $w_m(t)$:

$$\frac{d^2 w_n}{d\tau^2} + \omega_n^2 w_n - 4v^* \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^k \sin\left(\frac{\pi(2k+1)\tau}{2}\right) b_{nmk} w_m = 0, \quad n = 0,1,2,\ldots \tag{37}$$

$$b_{nmk} \parallel \varphi_n(z) \parallel^2 = \int_0^1 \varphi_n(z) \left[ \cos\left(\frac{\pi(2k+1)z}{2}\right) \frac{d\varphi_m(z)}{dz} + \frac{2\sin\left(\frac{\pi(2k+1)z}{2}\right)}{\pi(2k+1)} \frac{d^2\varphi_m(z)}{dz^2} \right] dz. \tag{38}$$

Here, $v^* = V_*/a$ is a relative impact velocity to be considered a minor parameter.

Let us estimate the parameters of the set (37)–(38) for two cases (see e.g. Bogolubov and Mitropolsky, 1970; Mitropolsky, 1967; Berezovsky, 1976).

1. Simple resonance. Assume, at some $i$ and $j$, the relation

$$\omega_j = \pi(2i+1)/2+\mu\vartheta; \quad \mu^2 = 4v^*, \tag{39}$$

is fulfilled, i.e., one of the frequencies of the longitudinal vibration is close to the frequency of the longitudinal vibrations. Hence the relation between the resonance frequencies and the relative bar rigidity, $\varsigma$, can be found:

$$\zeta = \frac{2q_i^2}{\dfrac{\pi(2i+1)}{2}+\mu\vartheta} = L\sqrt{\frac{S}{J}}. \tag{40}$$

The parameter $\vartheta$ defines the difference between these frequencies. It is supposed to be proportional to the minor parameter. It is also supposed that this relation is not fulfilled at any other $i$ and $j$.

Figure 8 shows the boundary of the stability region as dependent on the parameter $m$ at $i = 0$, $j = 3$. The lateral vibration frequency dependent on the relative rigidity $\varsigma$ is plotted on the horizontal axis, the parameter $m$ proportional to the square root of impact velocity is plotted on the vertical axis. The resonance frequency is marked by a thin vertical line. The regions where the real parts of the roots of the characteristic equation are positive are denoted by "$n$", whereas the regions with the zero parts are denoted by "$s$". Note that, because of calculation errors, the real parts of the eigenvalues may be positive, though very small (about the values of calculation errors).

2. Combinative resonance. Assume, at some $i$, $j$ and $m$, the relation

$$\omega_i+\omega_j+\varepsilon(\vartheta_1+\vartheta_2) = \pi(2m+1)/2 \tag{41}$$

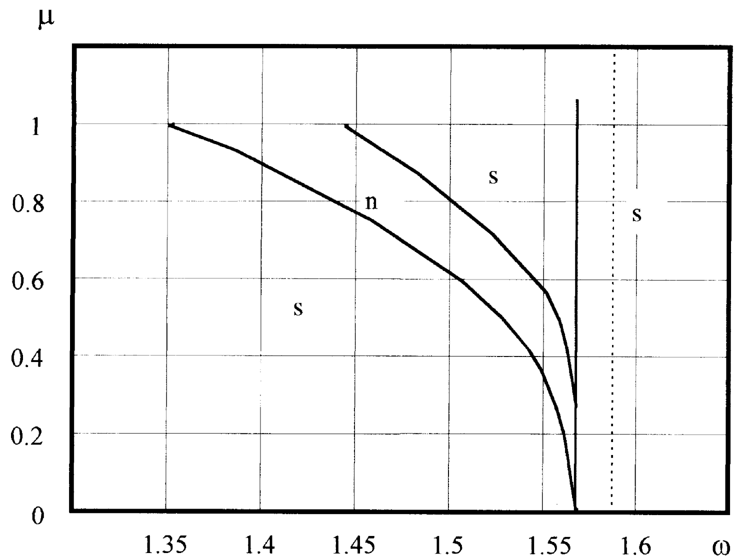


Fig. 8. Domains of stability for elastic bar rectilinearity.

is fulfilled. Also assumed that, for the rest of the frequency pairs, these relations are not fulfilled at any $i, j, m$, and at any $i$ and $m$ we have $w_i \neq p(2m+1)/2$.

Thus, the solutions given in Sections 1 and 2 are realizable in practice, as they are stable with respect to lateral vibrations at certain values of the parameters.

## 4. Conclusion

The examples discussed in the paper have shown good results in applying the method for solving boundary value problems of deformed solid mechanics described in Part 1 and Komogorov (1995). The method can be applied to the prediction of the strain–stress state and body fragmentation under macrofracture as well as to the study of motion stability described in Part 1 and Kolmogorov (1995). The solutions obtained by the method coincide with the well-known solutions and explain some experimental phenomena. The method can be recommended for practical application.

## Acknowledgements

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